# MAGNETOHYDRODYNAMIC BOUNDARY LAYER IN A MEDIUM WITH ANISOTROPIC CONDUCTIVITY FOR SMALL MAGNETIC REYNOLDS NUMBERS

(MAGNITOGIDRODINAMICHESKII POGRANICHNYI SLOI V SREDE S ANISOTROPNOI PROVIDOMOSTIU PRI MALYKH MAGNITNYKH CHISLAKH REYNOLDSA)

*PMM Vol.26, No.6, 1962, pp. 1077-1086*

G.A. LIUBIMOV
(Moscow)

*(Received June 28, 1962)*

**The problem of the magnetohydrodynamic boundary layer in an anisotropically conducting incompressible medium is formulated. Considerable use is made of the expressions obtained in [1]. An investigation is made of the boundary layer on a flat plate of dielectric material for small values of the parameter**

$$mL = \frac{\sigma H^2 L}{\rho c^2 U\,(1+\omega^2\tau^2)}$$

**The magnetic field is taken to be uniform and perpendicular to the plate. In contrast to the boundary layer in a medium with isotropic conductivity [2], the flow in a boundary layer with anisotropic conductivity is not plane, therefore the corresponding similarity problem does not reduce to one but to two ordinary differential equations.**

1. We shall investigate the problem of a boundary layer in an incompressible fluid having anisotropic conductivity. Let the magnetic Reynolds number be small

$$R_m = \frac{4\pi\sigma U L}{c^2} \ll 1$$

In that case, the induced currents are small, and it may be assumed that the magnetic field $\mathbf{H}$ is due to currents outside the fluid (for example, in the body about which the fluid is flowing). It follows that the magnetic field in the flow field satisfies the equation rot $\mathbf{H} = 0$, and the currents are determined from the generalized Ohm's law

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \alpha\,(\mathbf{j}\times\mathbf{H}) \qquad \left(\alpha = \frac{\omega\sigma}{H}\right) \qquad (1.1)$$

Let the conditions be such that the electrons have spiral paths ($\omega_e\tau_e \equiv \omega\tau \gtrsim 1$), but the ions do not ($\omega_i\tau_i \ll 1$). Under these conditions (we neglect gradients of the electron pressure), Ohm's law [3] has the form (1.1), and the coefficient of viscosity has the same value as it has in the given fluid in the absence of a magnetic field [4]. To simplify the formulas, we shall take the coefficient of viscosity $\nu$, the conductivity $\sigma$ and the parameter $\alpha$ to be constant.

From (1.1) we have

$$\mathbf{j} = \frac{\sigma}{1+\omega^2\tau^2}\left\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H} + \alpha\left[\mathbf{H}\times\mathbf{E} + \frac{1}{c}\mathbf{v}H^2 - \frac{1}{c}\mathbf{H}(\mathbf{v}\cdot\mathbf{H})\right] + \alpha^2\mathbf{H}(\mathbf{E}\cdot\mathbf{H})\right\} \tag{1.2}$$

Taking the divergence (div) of equation (1.2) and making use of Maxwell's equations and the condition rot $\mathbf{H} = 0$, we obtain the following equation, which determines the charge density in the flow field

$$4\pi\rho_e = -\left\{\frac{1}{c}\mathbf{H}\,\mathrm{rot}\,\mathbf{v} + \frac{2\alpha}{c}\,\mathbf{v}\,(\mathbf{H}\nabla)\,\mathbf{H} - \frac{\alpha}{c}\,[(\mathbf{H}\nabla)\,\mathbf{v} + (\mathbf{v}\nabla)\,\mathbf{H}]\cdot\mathbf{H} + \alpha^2\mathbf{H}\,[(\mathbf{E}\nabla)\,\mathbf{H} + (\mathbf{H}\nabla)\,\mathbf{E}]\right\} \tag{1.3}$$

The maximum electric field which can be created by charge separation in the flow field is, in order of magnitude, equal to the field with $\mathbf{j} = 0$, and thus, in view of (1.1), may be assumed to be $E_* \approx UH/c$.

We shall assume that external electric fields (due to sources distributed outside the flow field) do not exist, or, at least, are no greater than fields due to the charge distribution (1.3) in the flow field ($E \lesssim UH/c$). If strong electric fields are created by external sources $\rho_{e+}$ (for example, in flow over electrodes to which a high potential is applied), then these fields are either shielded by a surface charge (if the external conditions are such that current is not supplied to the fluid), or else there are created currents which are closed through an external circuit. Currents which flow in the fluid due to external fields may strongly affect the magnetic field in the fluid, $R_* \ll 1$ notwithstanding, and can produce an important force on the flow.

In the latter case, due to the linearity of the equations of electrodynamics, it is possible to calculate immediately the currents in the fluid due to external sources, taking the fluid to be at rest. These currents must be included in the evaluation of the magnetic field. Thereafter, it is necessary to solve the problem of the motion of the fluid and the distribution of the currents and fields outside the flow field, induced by the motion of the fluid.

In the solution of the hydrodynamic problem, the electromagnetic force has to be represented by two parts, one of which is connected with external currents and currents inside the fluid due to external electric fields (this part is known from the solution of the hydrodynamic problem), and a part connected with the induced currents.

$$\mathbf{j} = \rho_{e+}\mathbf{E}_+ + \rho_e\mathbf{E}_+ + \rho_{e+}\mathbf{E} + \frac{1}{c}\mathbf{j}_+ \times \mathbf{H} + \frac{1}{c}\mathbf{j} \times \mathbf{H}$$

$$4\pi\rho_{e+} \equiv -\alpha^2\mathbf{H}\,[(\mathbf{E}_+\nabla)\,\mathbf{H} + (\mathbf{H}\nabla)\,\mathbf{E}_+] \tag{1.4}$$

$$\mathbf{j}_+ = \frac{\sigma}{1+\omega^2\tau^2}\{\mathbf{E}_+ + \alpha\,(\mathbf{H}\times\mathbf{E}_+) + \alpha^2\mathbf{H}\,(\mathbf{E}_+\mathbf{H})\}$$

Here $\mathbf{E}_+$ and $\mathbf{H}$ are determined by Maxwell's equations from the given external sources, $\mathbf{E}$ is an unknown function which is determined from the solution of the magnetohydrodynamic problem, $\rho_e$ and $\mathbf{j}$ depend only on the induced part of the electric field $\mathbf{E}$, and are determined by $\mathbf{E}$, $\mathbf{H}$, $\mathbf{v}$ through the relations (1.3) and (1.2).

Since $E \lesssim UH/c$, and comparing the terms $\rho_e\mathbf{E}_+$ and $\rho_{e+}\mathbf{E}$ with $(\mathbf{j}_+ \times \mathbf{H})/c$ in (1.4), it is easy to establish that

$$|\rho_{e+}\mathbf{E}| \sim |\rho_e\mathbf{E}_+| \ll \frac{1}{c}|\mathbf{j}_+ \times \mathbf{H}| \tag{1.5}$$

if

$$\frac{U}{\sigma L}\max\{1+\omega\tau,\ \omega^2\tau^2(1+\omega^2\tau^2)\} \ll 1$$

In addition, $|\mathbf{j}| \ll |\mathbf{j}_+|$ for $|\mathbf{E}_+| \gg |\mathbf{E}|$. Thus, if the external fields are large, $E_+ \gg UH/c$, and due to these fields large currents can flow in the fluid, and if, in addition, $\omega\tau$ is not too large, so that (1.5) holds, then in the solution of the hydrodynamic problem

$$\mathbf{f} = \rho_{e+}\mathbf{E}_+ + \frac{1}{c}\mathbf{J}_+ \times \mathbf{H} \tag{1.6}$$

may be used instead of (1.4).

We may also note that the condition

$$\frac{U}{\sigma L} \sim \frac{U^2}{c^2}\frac{1}{R_m} \ll 1$$

is always fulfilled, since it is the condition for neglect of displacement currents.

The relation (1.6) shows that, for the conditions that have been laid down, forces are exerted on the fluid only by an external field and by currents which it produces. In this case, the hydrodynamic problem uncouples itself from the electric one. The electromagnetic force (1.6) in the magnetohydrodynamic equations will have a given value, which applies not only to the basic flow but also to the boundary layer. Having solved the hydrodynamic problem, it is possible to find the electric field **E** from Maxwell's equations using (1.3), and then the distribution of the induced currents from (1.2). For very large values of $\omega\tau$, for which the inequality (1.5) is violated, the decoupling does not occur. An example of an analogous formulation of the problem for flow in a canal in a strong electric field is given in [6].

The decoupled determination of the electric field of the external sources $\mathbf{E}_+$ and the sources induced by the motion of the fluid is significant for large external fields $E_+ \gg UH/c$, for which this approach gives a simplification of the problem. If external fields do not exist in the flow field, or if they do not exceed in order of magnitude the values of the induced field ($E_+ \lesssim UH/c$), then it makes sense, evidently, to solve the hydrodynamic and electromagnetic problems simultaneously for given external sources: in this case, for $R_m \ll 1$, currents in the fluid need not be included in computing the magnetic field, and the latter may be computed separately from the given external currents. Having the above remarks in mind, we shall now investigate problems in which the external electric fields are comparable to the induced ones*.

If the external electric fields are comparable to the induced ones, then $E \lesssim UH/c$ and $j \lesssim \sigma UH/c$, as follows from (1.2). The change of the magnetic field across the boundary layer is small, while the thickness of the boundary layer is determined by viscous forces [1]. From this it follows that inside the boundary layer

$$\partial \mathbf{H} / \partial y = 0 \tag{1.7}$$

Here, $y$ is the coordinate across the boundary layer. Therefore in the boundary layer equations $\mathbf{H}(x, z)$ is a function of the coordinates $x$, $z$ along the boundary layer. This function is the value of a function $\mathbf{H}^*(x, y, z)$ at points of the boundary of the body, i.e. $\mathbf{H}^*(x, o, z) = \mathbf{H}(x, z)$.

The function $\mathbf{H}^*$ satisfies the equations

$$\operatorname{rot} \mathbf{H}^* = \mathbf{j},^* \qquad \operatorname{div} \mathbf{H}^* = 0 \tag{1.8}$$

Here $\mathbf{j}^*$ is the current density outside the flow field.

* In connection with the possibility of separating the electric fields into external and induced currents, note should be made of a similar hypothesis made in [1].

In accordance with (1.3), the charge density inside the boundary layer is

$$\rho_e \lesssim \frac{UH}{c\delta} \tag{1.9}$$

($\delta$ is a characteristic thickness of the viscous boundary layer.) From this it follows that the "electric" force $\rho_e \mathbf{E}$ in the boundary layer is negligibly small in comparison with the "magnetic" force $(\mathbf{j} \times \mathbf{H})/c$ if

$$\frac{U}{\sigma\delta} \max \{1 + \omega^2\tau, \omega^2\tau^2 (1 + \omega^2\tau^2)\} \ll 1 \tag{1.10}$$

In cases of practical interest, this inequality may be violated [1] only for very large $\omega\tau$.

We note that, in view of (1.5), the electric force is also negligibly small, compared to the magnetic force, in the flow outside the boundary layer. It is interesting that, for increasing $\omega\tau$ (1.9) is first violated ($\delta \ll L$) and then (1.5), i.e. with increasing $\omega\tau$ the electric force first becomes effective in the boundary layer equations, and only for very large $\omega\tau$ does it become effective in the main flow (for such large $\omega\tau$ the condition $\omega_i\tau_i \ll 1$ may be violated).

Thus, we have for the electromagnetic force the expression

$$\mathbf{f} = \frac{1}{c}\mathbf{j}\times\mathbf{H} = \frac{\sigma}{c(1+\omega^2\tau^2)}\left\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H} + \alpha\left[\mathbf{H}\times\mathbf{E} + \frac{1}{c}\mathbf{v}H^2\right]\right\}\times\mathbf{H} \tag{1.11}$$

In order to write the expression for the electromagnetic force in the boundary layer equations, terms of order $\delta$ must be dropped in equation (1.11), since only terms of order unity are retained in the boundary layer equations. The expression for the main part of the electromagnetic force $\mathbf{f}^\circ$ figuring in the boundary layer equations is easily obtained, if the estimates of boundary layer theory are made in (1.11):

$$\mathbf{f}^\circ = \frac{1}{c}\mathbf{j}^\circ\times\mathbf{H} = \frac{\sigma}{1+\omega^2\tau^2}\left\{\mathbf{E}^\circ + \frac{1}{c}\mathbf{v}_\tau\times\mathbf{H} + \alpha\left[\mathbf{H}\times\mathbf{E}^\circ + \frac{1}{c}\mathbf{v}_\tau H^2\right]\right\}\times\mathbf{H} \tag{1.12}$$

Here $\mathbf{E}^\circ$ is the main part of the electric field [1], i.e. the electric field computed with an accuracy up to terms of order $\delta$. The index $\tau$ denotes a projection of the corresponding vector onto the $x$, $z$ plane.

In [1] it is shown that, if the charge density in the boundary layer satisfies condition (1.9), the tangential component of the main part of the electric field $\mathbf{E}_\tau^\circ$ satisfies, in the boundary layer, the equation

$$\partial\mathbf{E}_\tau^\circ / \partial y = 0 \tag{1.13}$$

while the normal component ($E_y^\circ$) is related to the charge density in the boundary layer by the equation

$$\partial E_y^\circ / \partial y = 4\pi\rho_e^\circ \tag{1.14}$$

Here $\rho_e^\circ$ is the charge density, with accuracy up to terms of order $UH/c\delta$.

Equation (1.13) shows that $\mathbf{E}_\tau^\circ$ does not change across the boundary layer. In the boundary layer equations, $\mathbf{E}_\tau^\circ$ is to be considered a known function of the coordinates $x$, $z$ determined from the solution to the problem outside the boundary layer (in problems of flow over bodies, this corresponds to the solution of the flow problem in the basic flow and the distribution of the field inside the body). Equation (1.14) gives the variation of $E_y^\circ$ in the boundary layer.

In the problem under consideration, the charge density in the boundary layer is given by equation (1.3). It is easy to prove, in this case, taking into account (1.7) and (1.13), that $\rho_e^\circ$ satisfies the relation

$$4\pi\rho_e^\circ = -\frac{1}{c}\frac{\partial}{\partial y}|\mathbf{v}_\tau \times \mathbf{H}_\tau| + \frac{\alpha}{c}H_y\frac{\partial}{\partial y}(\mathbf{v}_\tau\cdot\mathbf{H}_\tau) - \alpha^2 H_y^2\frac{\partial E_y^\circ}{\partial y} \tag{1.15}$$

Integrating (1.14), making use of (1.15), we obtain

$$E_y^\circ = \frac{1}{1+\alpha^2 H_y^2}\left[-\frac{1}{c}|\mathbf{v}_\tau\times\mathbf{H}_\tau| + \frac{\alpha H_y}{c}(\mathbf{v}_\tau\cdot\mathbf{H}_\tau)\right] + \varphi(x, z) \tag{1.16}$$

Here $\varphi(x, z)$ is an arbitrary function.

Considering the boundary layer on the body, $\mathbf{v}_\tau = 0$ for $y = 0$, therefore $E_y^\circ|_{y=0} = \varphi(x, z)$ on the body. If the body is a dielectric, then $j_n = 0$ for $y = 0$ and, therefore, in view of (1.2)

$$\varphi(x, z) = -\frac{1}{1+\alpha^2 H_y^2}[\alpha\,|\mathbf{H}_\tau\times\mathbf{E}_\tau^\circ| + \alpha^2 H_y \mathbf{E}_\tau^\circ\cdot\mathbf{H}_\tau]_{y=0} \tag{1.17}$$

In this case, (1.16) gives the distribution of $E_y^\circ$ in the boundary layer, while for the solution of the problem outside the boundary layer the following boundary condition is obtained from (1.16):

$$E_y^\circ|_{y=\infty} = E^*{}_y|_{y=0} = \frac{1}{1+\alpha^2 H_y^2}\left[-\frac{1}{c}|\mathbf{U}\times\mathbf{H}_\tau| + \right. \tag{1.18}$$
$$\left. + \frac{\alpha H_y}{c}(\mathbf{U}\cdot\mathbf{H}_\tau) - \alpha\,|\mathbf{H}_\tau\times\mathbf{E}_\tau^*| - \alpha H_y^2 \mathbf{E}_\tau^*\cdot\mathbf{H}_\tau\right]_{y=0}$$

Here $\mathbf{U}$ is the velocity of the flow outside the boundary layer, the asterisk denotes the field which is determined in the solution of the problem of flow of an ideal fluid and the problem of the distribution of the field in the body.

If the body is a conductor, the boundary condition for $\mathbf{E}^*$ will be, from (1.13)

$$\mathbf{E}_\tau^* = 0 \tag{1.19}$$

In this case, $\varphi(x, z)$ in (1.16) is determined by the solution of the problem outside the boundary layer

$$\varphi(x, z) = E_y^*|_{y=0} - \frac{1}{1+\alpha^2 H_y^2}\left[-\frac{1}{c}[\mathbf{U}\times\mathbf{H}_\tau] + \frac{\alpha H_y}{c}\mathbf{U}\cdot\mathbf{H}_\tau\right]_{y=0} \tag{1.20}$$

and the function $\varphi(x, z)$ is related to the current density $j_y^\circ$ by the relation

$$\varphi(x, z) = \frac{1+\omega^2\tau^2}{1+\alpha^2 H_y^2}\, j_y^\circ \tag{1.21}$$

as follows from (1.2). Equations (1.2), (1.13) to (1.21) show that, for the main part of the normal component of the current density in the boundary layer, the following relation is valid

$$\partial j_y^\circ / \partial y = 0 \tag{1.22}$$

From (1.16) to (1.18) and (1.20) it follows that $E_y^\circ$ does not vary across the boundary layer ($\partial E_y^\circ/\partial y = 0$) only if $\mathbf{H}_\tau = 0$. The corresponding boundary layer equations for that case were obtained in a recently published paper [7]. From (1.12) it follows that the change of pressure across the boundary layer is of order $\delta$.

The boundary layer problem may be formulated as follows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \nu\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho} f_x^\circ \qquad \frac{\partial p}{\partial y} = 0, \tag{1.23}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - \nu\frac{\partial^2 w}{\partial y^2} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho} f_z^\circ, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Here $\mathbf{f}^\circ$ is determined by equation (1.12), in which $E_y^\circ$ is given by (1.16), while $\mathbf{E}_\tau^\circ$, $\varphi(x, z)$ and $\mathbf{H}(x, z)$ are determined from the boundary conditions and the solution of the external problem.

We may note that the system (1.23) remains the same for $R_m \lesssim 1$, for which case it changes only in the formulation of the outer problem. The formulation of the outer problem for $R_m \lesssim 1$ is analogous [1] to its formulation for $\omega\tau = 0$, only the expression for the magnetic force changes, which for $\omega\tau \neq 0$ must be given by expression (1.11).

**2. Boundary layer on a half-infinite flat plate.** As an example, we shall investigate the problem of the boundary layer on a semi-infinite, dielectric flat plate.

We shall take the plate to be the half-plane $y = 0$, $x > 0$. We shall assume that the magnetic field in the region $x > 0$ is uniform and normal to the plate ($\mathbf{H} = H_0\mathbf{e}_y$); in the region $x < 0$ we have $\mathbf{H} = 0$. In the region $x < 0$ the flow does not interact with a magnetic field and it may be considered to be a uniform flow with velocity $U$ along the $x$-axis.

In addition, we shall assume that all quantities are independent of $z$, i.e. $\partial/\partial z = 0$. This assumption becomes obvious if the problem of the flow over the plate is considered to be the limiting case of flow over a body of revolution. With this assumption, it follows from the equation rot $\mathbf{E} = 0$ and the condition that the electric field vanish at infinity, that $E_z = 0$.

With the assumptions made, equations (1.23) take the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{H_0^2\sigma}{c^2\rho(1+\omega^2\tau^2)}\left\{u + \omega\tau w - \omega\tau\frac{cE_x^\circ}{H_0}\right\}$$
$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} - \nu\frac{\partial^2 w}{\partial y^2} = \frac{H_0^2\sigma}{c^2\rho(1+\omega^2\tau^2)}\left\{\omega\tau u - w + \frac{cE_x^\circ}{H_0}\right\} \qquad (2.1)$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad E_x^\circ = E_x^*(x, 0, z), \qquad p = p^*(xz)$$

The plate is a dielectric and $H_\tau = 0$, therefore it follows from (1.16) to (1.18) that $E_y^\circ = \varphi(x, z) = 0$ in the boundary layer, while $E_x^\circ$ is determined from the solution of the outer problem.

Using (1.2), (1.3) and (1.11), the equations describing the flow outside the boundary layer may, with the assumptions made, be written in the form

$$u^*\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} - \frac{H_0^2\sigma}{c^2\rho(1+\omega^2\tau^2)}\left\{u^* + \omega\tau w^* - \omega\tau\frac{cE_x^*}{H_0}\right\}$$
$$u^*\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y}$$
$$u^*\frac{\partial w^*}{\partial x} + v^*\frac{\partial w^*}{\partial y} = \frac{H_0^2\sigma}{c^2\rho(1+\omega^2\tau^2)}\left\{\omega\tau u^* - w^* + \frac{cE_x^*}{H_0}\right\}, \quad \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0 \qquad (2.2)$$
$$\text{rot}\,\mathbf{E}^* = 0, \qquad \text{div}\,\mathbf{E}^* = 4\pi\rho_e^* = \frac{\partial w^*}{\partial x} + \omega\tau\frac{\partial v^*}{\partial y} - \omega^2\tau^2\frac{\partial E_y^*}{\partial y}$$

The asterisks denote quantities connected with the outer flow. From (1.18), the solution of the system (2.2) describing the outer flow must satisfy the following boundary conditions:

$$v^* = 0, \qquad E_y^* = 0 \qquad \text{for } y = 0 \qquad (2.3)$$

In addition, the form of the solution of the system (2.2) depends on the conditions at infinity, in particular on the behavior of the electric field at infinity, but these conditions are connected with the particulars of the problem, and have to be formulated in the statement of the problem.

The system of equations (2.2) has a solution corresponding to uniform flow $u^* = U$, $v^* = w^* = 0$ only for $j_x = 0$, when charge accumulates at infinity to balance the Hall field in the $x$-direction. The solution of the problem for that case is discussed in [7].

In what follows we shall assume that the charge which is carried to infinity by the currents $j_x$ is neutralized in some manner, i.e. $j_x \neq 0$. Naturally, the solution of the problem will depend on the conditions for the electric field at infinity (the conditions for neutralizing the charge).

To simplify the solution of the problem, we shall assume that the parameter

$$mL = \frac{H_0^2 \sigma L}{c^2 \rho U (1 + \omega^2 \tau^2)} \ll 1 \tag{2.4}$$

i.e. the electromagnetic action on the flow is small. In that case the problem in the boundary layer and in the outer flow may be linearized around the Blasius solution (an analogous treatment of the problem for $\omega\tau = 0$ is given in [2]). We shall assume that $\omega\tau$ is not too large, so that

$$\omega\tau mL \ll 1$$

Let us write all quantities in the form

$$u = u_0 + mLu_1, \qquad v = v_0 + mLv_1, \qquad E = E_0 + mLE_1 \quad \text{etc.} \tag{2.5}$$

The subscript zero denotes quantities corresponding to the solution for $mL = 0$ ($\sigma = 0$). Introduce the dimensionless quantities

$$x \equiv \frac{x}{L}, \quad y \equiv \frac{y}{L}, \quad u \equiv \frac{u}{U}, \quad v \equiv \frac{v}{U}, \quad w \equiv \frac{w}{U}$$
$$p \equiv \frac{p}{\rho U^2}, \quad \mathbf{E} \equiv \frac{c\mathbf{E}}{UH_0}, \quad R = \frac{UL}{\nu} \tag{2.6}$$

Here $U$ is the velocity of the outer flow corresponding to the solution of the problem for $mL = 0$ ($U = u_0^*$), $R$ is the Reynolds number. In what follows, the dimensionless quantities (2.6) are used throughout.

Putting (2.5) into equations (2.1) and (2.2) and collecting the main terms (not containing $mL$) we find that quantities with index zero in the boundary layer correspond to the Blasius problem, and quantities with index zero in the outer flow correspond to a uniform flow with velocity $u_0^* = U$.

To determine the quantity $\mathbf{E}_0^*$ we obtain from (2.2) the equations

$$\operatorname{rot} \mathbf{E}_0^* = 0, \qquad \operatorname{div} \mathbf{E}_0^* = -\omega^2\tau^2 \frac{\partial E_{y_0}^*}{\partial y}$$

We shall assume that charge cannot accumulate at infinity (complete charge neutralization at infinity). Then the electric field potential $\Phi(\mathbf{E} = \operatorname{grad} \Phi)$ is equal to zero at infinity. Since, in addition, the condition $E_y^* = \partial\Phi/\partial y = 0$ holds on the plate $(y = 0)$, then $\mathbf{E}_0^* = 0$ is a valid solution for $\mathbf{E}_0^*$. We may note that, with the assumptions adopted, there is a current along the outer flow whose density, to terms of order $mL$, is given by $j_x = \sigma\omega\tau UH_0/c(1 + \omega^2\tau^2)$.

Using the solution for $mL = 0$, it is possible to obtain from systems (2.1), (2.2) linear systems of equations for the corrections ($u_1$, $v_1$, $\mathbf{E}_1$, etc.) to the solution for $mL = 0$, which are connected with the effect of the electro-magnetic forces. The system of equations for quantities connected with the outer flow has the form

$$\frac{\partial u_1^*}{\partial x} = -\frac{\partial p_1^*}{\partial x} - 1, \qquad \frac{\partial v_1^*}{\partial x} = -\frac{\partial p_1^*}{\partial y}, \qquad \frac{\partial w_1^*}{\partial x} = \omega\tau, \qquad \frac{\partial u_1^*}{\partial x} + \frac{\partial v_1^*}{\partial y} = 0$$
$$\operatorname{rot} \mathbf{E}_1^* = 0, \qquad \operatorname{div} \mathbf{E}_1^* = \frac{\partial w_1^*}{\partial x} + \omega\tau\frac{\partial v_1^*}{\partial y} - \omega^2\tau^2\frac{\partial E_{y1}^*}{\partial y} \tag{2.7}$$

and for quantities relating to the boundary layer the form

$$u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_0}{\partial y} - \frac{1}{R}\frac{\partial^2 u_1}{\partial y^2} = -\frac{\partial p_1}{\partial x} - u_0 \tag{2.8}$$
$$u_0\frac{\partial w_1}{\partial x} + v_0\frac{\partial w_1}{\partial y} - \frac{1}{R}\frac{\partial^2 w_1}{\partial y^2} = \omega\tau u_0, \qquad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \qquad p_1 = p_1^*(xz)$$

We may note that if (2.4) is valid but (2.5) is violated due to large $\omega\tau$, then a term $\omega\tau w_1$ will appear in the first equations of (2.7) and (2.8).

Since there are currents flowing in the flow outside the boundary layer, this flow, generally speaking, will not be uniform, due to the action of electromagnetic forces. In the approximation being considered, the projection of the electromagnetic force on the direction of the basic flow ($x$-axis) is constant. Therefore, by having a constant pressure gradient $(\partial p_1^*/\partial x \neq 0)$, it is possible to obtain a velocity in the $x$-direction which is constant and equal to the velocity of the outer flow for $mL = 0$ $(u_1^* = 0,\ u^* = u_0^* + mLu_1^* = U)$. Here the quantity $\partial p_1^*/\partial x$ has to be determined from the solution of the system (2.7). We note that, due to the assumption $\partial p_1^*/\partial z = 0$, the component of electromagnetic force in the $z$-direction, which in the given approximation is also constant, cannot be balanced by a pressure gradient and, therefore, there is a $z$-component of velocity in the outer flow. Since the component of electromagnetic force in the $y$-direction in the outer flow is zero, it is natural to take $\partial p^*/\partial y = 0$.

With the assumptions made, system (2.7) has the solution

$$u_1^* = 0, \qquad v_1^* = 0, \qquad \partial p_1^* / \partial x = -1, \qquad w_1^* = \omega\tau x \tag{2.9}$$

for the dynamic quantities.

For determining the perturbations $\mathbf{E}_1^*$ to the electric field, the following system of equations is obtained from (2.7) together with (2.9):

$$\operatorname{rot} \mathbf{E}_1^* = 0, \qquad \operatorname{div} \mathbf{E}_1^* = \omega\tau - \omega^2\tau^2 \partial E_{y1}^* / \partial y$$

This system has to be solved in the context of the assumptions made above, taking into account that the electric potential $\Phi_1 (\mathbf{E}_1^* = \operatorname{grad} \Phi_1)$ is equal to zero at infinity and $\partial \Phi_1/\partial y = 0$ at $y = 0$.

With (2.9), the system (2.8) can finally be put in the form

$$u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} - \frac{1}{R} \frac{\partial^2 u_1}{\partial y^2} = 1 - u_0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad u_0 \frac{\partial w_1}{\partial x} + v_0 \frac{\partial w_1}{\partial y} - \frac{1}{R} \frac{\partial^2 w_1}{\partial y^2} = \omega\tau u_0 \tag{2.10}$$

In virtue of (2.9), the boundary conditions have the form

$$u_1 = v_1 = w_1 = 0 \quad \text{for } y = 0, \qquad u_1 = 0, \;\; w_1 = \omega\tau x \text{ for } y = \infty \tag{2.11}$$

It is easy to see that the first two equations of the system (2.10) do not contain $w_1$, and, therefore, can be integrated separately. The solution of these two equations for the boundary conditions (2.11) may be reduced to the solution of a single differential equation in the variable $\eta = y\sqrt{(R/x)}$, which is integrated in [2]. It is evident that in the linearized case the projection of the velocity on the $xy$-plane depends on $\omega\tau$ only through the parameter $mL$.

The last equation of (2.10) can be reduced to an ordinary differential equation if the function $\Psi(\eta)$ is introduced through the following relation

$$w_1 = \omega\tau x \Psi(\eta) \tag{2.12}$$

Then the last equation of (2.10) takes the form

$$\Psi'' + \tfrac{1}{2} f_0 \Psi' - f_0' \Psi + f_0' = 0 \qquad (2.13)$$

Here $f_0(\eta)$ is a function connected with the stream function of the Blasius solution by the relation

$$f_0(\eta) = \frac{\psi(xy)}{\sqrt{U\nu x}}$$

The boundary conditions (2.11) for $w_1$ give for $\Psi$ the form

$$\Psi(0) = 0, \qquad \Psi(\infty) = 1 \quad (2.14)$$

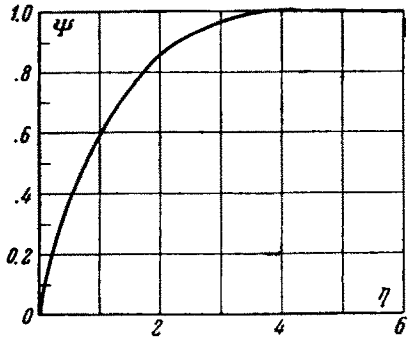

Fig. 1.

Equation (2.13) with the conditions (2.14) was numerically integrated. The function $\Psi(\eta)$ obtained is shown in Fig. 1.

We note that if (2.5) is violated then the system analogous to (2.10) will not reduce to a system of ordinary differential equations in the variable $\eta$.

Figures 2 and 3 show the profiles of the longitudinal velocity $u$ and the transverse velocity $w$ for different values of the parameter $\omega\tau$ at a fixed value of the parameter

$$m^* x = \frac{\sigma H_0^2 x}{\rho c^2 U} = 0.5$$

Figure 2 shows that the retardation of the flow in the $x$-direction due to electromagnetic forces is decreased with increasing $\omega\tau$. For $\omega\tau \to \infty$ (inequality (2.5) will be satisfied if $m^* x < 1$) the velocity profile in that direction tends toward the velocity profile for $H_0 = 0$.

Figure 3 shows that the transverse velocity grows at first with increasing $\omega\tau$ and then decreases, $w_{max}$ being reached for $\omega\tau = 1$. For $\omega\tau \to \infty$ the transverse velocity $w \to 0$.

We note that the approach to the Blasius solution for $\omega\tau \to \infty$ and fixed $\sigma$ is related to a decrease of the effective conductivity. From (1.1) it follows that, for $\omega\tau \to \infty$ and $\sigma$ = const, the currents become parallel to the magnetic field $\mathbf{j} \times \mathbf{H} \to 0$, i.e. in our case $j_x \to 0$, $j_z \to 0$ and, therefore, the electromagnetic action on the flow disappears.

The coefficients of the longitudinal and transverse resistances, up to terms of order $mL$ and expressed in dimensional quantities, have the form

$$c_x = 2\nu U^{-2}\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{2}{\sqrt{R_x}}\left(f_0'(0) + \frac{m^*x}{1+\omega^2\tau^2}f'_2(0)\right) =$$
$$= \frac{2}{\sqrt{R_x}}\left(0.332 + 1.147\frac{m^*x}{1+\omega^2\tau^2}\right)$$
$$c_z = 2\nu U^{-2}\frac{\partial w}{\partial y}\bigg|_{y=0} = \frac{2}{\sqrt{R_x}}m^*x\frac{\omega\tau}{1+\omega^2\tau^2}\Psi'(0) = \frac{2}{\sqrt{R_x}}\,1.35\;\; m^*x\frac{\omega\tau}{1+\omega^2\tau^2}$$

Here $R_x = Ux/\nu$ is the Reynolds number, $f_2(\eta)$ is a function related to $u$ by the relation $u = U(f_0' + mxf_2')$.

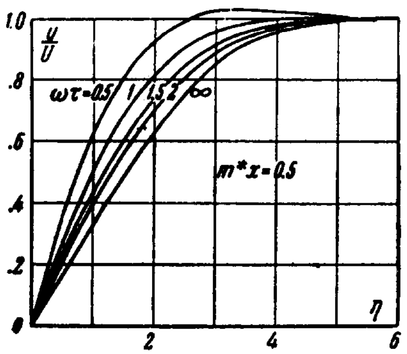


Fig. 2.

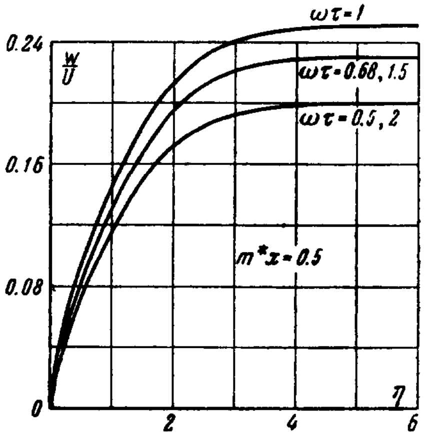


Fig. 3.

BIBLIOGRAPHY

1. Liubimov, G.A., K postanovke zadachi o magnitogidrodinamicheskom pogranichnom sloe (On the formulation of the problem of the magnetohydrodynamic boundary layer). *PMM*, Vol. 26, No. 5, 1962.

2. Rossow, V.U., On flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field. NACA TN 3971, 1957.

3. Liubimov, G.A., O forme zakona Oma v magnitnoi gidrodinamike (On the form of Ohm's law in magnetohydrodynamics). *PMM* Vol. 25, No. 4, 1961.

4. Gubanov, A.I. and Lun'kin, Iu.P., Uravneniia magnitnoi plasmodinamiki (The equations of magnetoplasmadynamics). *ZhTF* Vol. 30, No.9, 1960.

5. Kulikovskii, A.G. and Liubimov, G.A., *Magnitnaia gidrodinamika* (*Magnetohydrodynamics*). Fizmatgiz, 1962.

6. Liubimov, G.A., O reshenii nekotor'ikh zadach magnitnoi gidrodinamiki pri anizotropnoi providomosti (On the solution of certain problems of magnetohydrodynamics for anisotropic conductivity). *PMM* Vol.26, No. 3, 1962.

7. Gubanov, A.I. and Pushkarev, O.I., Viazkii pogranichn'ii sloi v magnitnoi gidrodinamiki pri konechnom $\omega\tau$ (The viscous boundary layer in magnetohydrodynamics for finite $\omega\tau$). *ZhTF* Vol. 32, No. 6, 1962.

*Translated by* A.R.